Föreläsning 28/11-13

<u>Stationary distribution</u> for birth and death process, i.e., try to solve $\pi G = 0$ $(\cancel{\pi B = \pi})$

$$\begin{cases} -\lambda_0 \pi_0 + \mu_1 \pi_1 = 0 \\ \lambda_{i-1}\pi_{i-1} - (\lambda_i + \mu_i)\pi_i + \mu_{i+1}\pi_{i+1} = 0 \quad , \quad i \geq 1 \end{cases}$$

$$\pi_i = \frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_i}\pi_0$$ solution?

Check: $-\lambda_0\pi_0 + \mu_1\pi_1 \Longrightarrow -\lambda_0\pi_0 = \mu_1 \frac{\lambda_0}{\mu_1}\pi_0 = 0$ OK!

$$\frac{\lambda_0 \cdots \cancel{\lambda_{i-2}}}{\mu_1 \cdots \cancel{\mu_{i-1}}}\pi_0 - \underbrace{(\lambda_i + \mu_i)\frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_i}\pi_0}_{\left(\cancel{\frac{\lambda_0 \cdots \lambda_i}{\mu_1 \cdots \mu_i}} + \cancel{\frac{\lambda_0 \cdots \lambda_{i-1}}{\mu_1 \cdots \mu_{i-1}}}\right)\pi_0} + \cancel{\frac{\lambda_0 \cdots \lambda_i}{\mu_1 \cdots \mu_i}}\pi_0 = 0 \quad \text{OK!}$$

<u>Poisson process</u>

$\mu^{(0)} = (1\ 0\ 0\ \ldots)$ $\quad \mu^{(t)} = \mu^{(0)} P_t \quad P_t = e^{tG} \quad P_t' = Ge^{tG} = GP_t = P_t G$

$$\mu^{(t)\prime} = (\mu^{(0)} P_t)' = \mu^{(0)} P_t' = \underbrace{\mu^{(0)} P_t}_{\mu(t)} G = \mu^{(t)} G$$

$$\mu^{(t)} = (\mu_0(t)\ \ \mu_1(t)\ \ \mu_2(t) \ldots) = (P(X(t)=0)\ P(X(t)=1) \ldots)$$

$$\begin{cases} \mu_0'(t) = -\lambda_0 \mu_0(t) \Longrightarrow \mu_0(t) = e^{-\lambda_0 t} + C = e^{-\lambda_0 t} \\ \mu_n'(t) = \lambda_0 \mu_{n-1}(t) - \lambda_0 \mu_n(t) \quad , \quad n \geq 1 \end{cases}$$

claim $\mu_n(t) = \frac{(\lambda_0 t)^n}{n!} e^{-\lambda_0 t}$

Check: $\mu_n(t)' = \frac{(\lambda_0 t)^{n-1}}{(n-1)!}\lambda_0 e^{-\lambda_0 t} - \lambda_0 \frac{(\lambda_0 t)^n}{n!} e^{-\lambda_0 t} = \lambda_0 \mu_{n-1}(t) - \lambda_0 \mu_n(t)$

OK!

<u>Birth process</u>

Same problem but with λ_i's being different...

$\mu^{(0)} = (1\ 0\ 0\ \ldots)$ $\quad \mu^{(t)} = ?$

Same argument: $\mu_0'(t) = -\lambda_0 \mu_0(t)$

$\mu_n'(t) = \lambda_{n-1}\mu_{n-1}(t) - \lambda_n \mu_n(t) \quad , \quad n \geq 1$

Use Fouriertransform!

Laplace: $f(t) \quad \hat{f}(\theta) = \int_0^\infty e^{-\theta t} f(t)\,dt$

$$f'(t) \quad \int_0^\infty e^{-\theta t} f'(t)\,dt = \left[e^{-\theta t} f(t)\right]_0^\infty + \theta \int_0^\infty e^{-\theta t} f(t)\,dt = \theta \hat{f}(\theta) - f(0)$$

$$\begin{cases} \theta \hat{\mu}_0(\theta) - \underbrace{\mu_0(0)}_{=1} = -\lambda_0 \hat{\mu}_0(\theta) \\ \theta \hat{\mu}_n(\theta) - \underbrace{\mu_n(0)}_{=0} = \lambda_{n-1} \hat{\mu}_{n-1}(\theta) - \lambda_n \hat{\mu}_n(\theta) \quad n \geq 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \hat{\mu}_0(\theta) = 1/(\lambda_0 + \theta) \\ \hat{\mu}_n(\theta) = \frac{\lambda_{n-1}}{\lambda_n + \theta} \hat{\mu}_{n-1}(\theta) = \ldots \text{iterate} \ldots = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{(\lambda_n+\theta)(\lambda_{n-1}+\theta)\ldots(\lambda_0+\theta)} = \end{cases}$$

$$= \{\text{partial fraction}\} = \frac{a_n}{\lambda_n+\theta} + \frac{a_{n-1}}{\lambda_{n-1}+\theta} + \cdots + \frac{a_0}{\lambda_0+\theta}$$

<u>Birth and death process</u>

Same problem, look at thm 6.11.10

<u>Analysis and processing of random variables</u> (chapter 6 in Hsu)

- Continuity, derivatives and integrals of random process.
- We look closer at properties of autocorrelation function $R_X(s,t) = E(X(s)X(t))$
- We look at PSD = powerspectraldensity = Fouriertr. of $R_X(T) = E(X(t)X(t+T))$ for WSS $X(t)$

- White noise = what is that?
- LTI = linear Invariant systems = "filter"

$$\underset{\text{in}}{X(t)} \longrightarrow \boxed{\text{LTI system}} \longrightarrow \underset{\text{out}}{Y(t)} \qquad Y(t) = \begin{cases} \int_{-\infty}^{\infty} h(t-u)X(u)du \\ \sum_{k=-\infty}^{\infty} h(n-k)X(k) \end{cases}$$

What is continuity, derivative and integral of process?

Continuity: $X(t)$ continous at time t if $X(t+\varepsilon) \longrightarrow X(t)$ as $\varepsilon \longrightarrow 0$.

$$\Longleftrightarrow \lim_{\varepsilon \to 0} E((X(t+\varepsilon) - X(t))^2) = 0$$

Derivative: $X(t)$ is differentiable at time t with derivative $X'(t)$ if

$$\frac{X(t+\varepsilon) - X(t)}{\varepsilon} \longrightarrow X'(t) \text{ as } \varepsilon \longrightarrow \infty$$

$$\Longleftrightarrow \lim_{\varepsilon \to 0} E\left(\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t)\right)^2\right) = 0$$

Integral: $\int_a^b X(t)dt \longleftarrow \sum_{i=1}^{n} X(\xi_i)(t_i - t_{i-1})$ where $a = t_0 < t_1 < \ldots < t_n = b$

$\xi_i \in [t_{i-1}, t_i]$

$\max_{1 \leq i \leq n} t_i - t_{i-1} \to 0$

ex. (continous)

When is WSS process $X(t)$ with autocorrelation function $R_X(\tau) = E(X(t)X(t+\tau))$ continous?

Solution:

$E((X(t+\varepsilon) - X(t))^2) = R_X(t+\varepsilon, t+\varepsilon) - 2R_X(t, t+\varepsilon) + R_X(t,t) =$

$2(R_X(0) - R_X(\varepsilon))$ Thus $X(t)$ continous iff $R_X(\tau)$ continous at $\tau = 0$.

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ex. (derivative)

If $X(t)$ is differentiable with autocorr. $R_X(s,t)$, what is $R_{X'}(s,t)$?

Solution:

$$R_{X'}(s,t) = E(X'(s)X'(t)) = E\left(\lim_{\varepsilon\to 0} \frac{X(s+\varepsilon)-X(s)}{\varepsilon} \frac{X(t+\varepsilon)-X(t)}{\varepsilon}\right) =$$

$$= \lim_{\varepsilon\to 0} E\left(\frac{X(s+\varepsilon)-X(s)}{\varepsilon} \frac{X(t+\varepsilon)-X(t)}{\varepsilon}\right) =$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{R_X(s+\varepsilon, t+\varepsilon) - R_X(s+\varepsilon, t)}{\varepsilon} - \frac{R_X(s, t+\varepsilon) - R_X(s,t)}{\varepsilon}}{\varepsilon} =$$

$$= \lim_{\varepsilon\to 0} \frac{\frac{\partial}{\partial t} R_X(s+\varepsilon, t) - \frac{\partial}{\partial t} R_X(s,t)}{\varepsilon} = \frac{\partial^2}{\partial s \partial t} R_X(s,t)$$

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ex. (integral)

$$E\left(\int_a^b X(s)\,ds \int_c^d Y(t)\,dt\right) = \int_a^b \int_c^d E(X(s)Y(t))\,ds\,dt =$$

$$= \int_a^b \int_c^d R_{X,Y}(s,t)\,ds\,dt.$$

Section 6.3

(more about autocorrelation fcn for WSS process $X(t)$)

$E(X(t)) = \mu$ constant, do not depend on t.

$R_X(t, t+\tau) = E(X(t)X(t+\tau)) = R_X(\tau)$ do not depend on t.

$$\begin{cases} R_X(-\tau) = R_X(\tau) & ① \\ R_X(0) \geq |R_X(\tau)| & ② \\ E(X(t)^2) = R_X(0) & ③ \end{cases}$$

Proof: ③ is trivial (by inspection)

① $R_X(-\tau) = E(X(t)X(t-\tau)) = E(X(t-\tau)X(t)) = R_X(\tau)$.

② $0 \leq E((X(t) \pm X(0))^2) = R_X(0) \pm 2R_X(t) + R_X(0) \Longrightarrow R_X(0) \geq \pm R_X(t)$ ▨